Integration of Rational Expressions by Partial Fractions

INTRODUCTION:

We start with a few definitions. A **rational expression** is formed when a polynomial is divided by another polynomial. In a **proper rational expression** the degree of the numerator is less than the degree of the denominator. In an **improper rational expression** the degree of the numerator is greater than or equal to the degree of the denominator.

This set of notes is given in three parts. Part A is an explanation of how to decompose a proper rational expression into a sum of simpler fractions. Part B explains Integration by Partial Fractions of proper rational expressions. Part C explains Integration by Partial Fractions of improper rational expressions. Each part includes detailed examples and a set of exercises.

PART A: Partial Fraction Decomposition

In mathematics we often combine two or more rational expressions into one.

E.g. $$\frac{4}{x+1}+\frac{3}{x-2} = \frac{4(x-2)}{(x+1)(x-2)}+\frac{3(x+1)}{(x-2)(x+1)} = \frac{7x-5}{(x-2)(x+1)}$$

Occasionally, however, the reverse procedure is necessary. The problem is to take a fraction whose denominator is a product of factors, and split it into a sum of simpler fractions. There is more than one way to do this. For the types of expressions we are dealing with the method illustrated here is probably the easiest to apply and understand (no system of equations to solve).

CASE 1 The denominator is a product of distinct linear factors.

For each distinct factor $ax+b$ the sum of partial fractions includes a term of the form $\frac{A}{ax+b}$.

Example 1 Rewrite $\frac{x+5}{(x-4)(x-1)}$ as a sum of simpler fractions.

First write the fraction as $\frac{x+5}{(x-4)(x-1)} = \frac{A}{x-4}+\frac{B}{x-1}$.

Multiply both sides by the common denominator to get $x+5 = A(x-1)+B(x-4)$. In this linear equation we can substitute any x-values to solve for A and B. However, the process is simpler if we choose those values which make a factor zero (i.e. $x-1=0$ or $x-4=0$).

Substitute $x=1$ in the linear equation. $1+5 = A(1-1)+B(1-4) \rightarrow 6=-3B \rightarrow B=-2$

Substitute $x=4$ in the linear equation. $4+5 = A(4-1)+B(4-4) \rightarrow 9=3A \rightarrow A=3$

Therefore, the partial fraction decomposition is $\frac{x+5}{(x-4)(x-1)} = \frac{3}{x-4}-\frac{2}{x-1}$.

First write the fraction as $\dfrac{x^2-2x-5}{x^3-5x^2} = \dfrac{x^2-2x-5}{x^2(x-5)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-5}$.

Multiply by $x^2(x-5)$ to get $x^2-2x-5 = Ax(x-5) + B(x-5) + Cx^2$.

Substitute $x=0$. $-5 = 0 + B(-5) + 0 \;\rightarrow\; B = 1$

Substitute $x=5$. $10 = 0 + 0 + C(25) \;\rightarrow\; C = \frac{2}{5}$

Substitute $x=1$. $-6 = A(-4) + (-4) + \frac{2}{5}(1) \;\rightarrow\; A = \frac{3}{5}$

The partial fraction decomposition is $\dfrac{{}^{2}\quad 2 \quad 5}{{}^{3}\quad 5\quad {}^{2}} \quad \dfrac{3}{5}\,\dfrac{1}{\;} \quad \dfrac{1}{{}^{2}} \quad \dfrac{2}{5}\cdot\dfrac{1}{-5} \quad \cdot \quad \cdot \quad =$

Substitute $x=0$ (and $A=-\frac{1}{2}$). $-3 = -\frac{1}{2}(3) + (C)(1) \rightarrow C = -\frac{3}{2}$

Substitute $x=1$ (and $A=-\frac{1}{2}$ and $C=-\frac{3}{2}$). $-1 = -\frac{1}{2}(2) + \left(B-\frac{3}{2}\right)(2) \rightarrow B = \frac{3}{2}$

The partial fraction decomposition is $\dfrac{x^2+x-3}{(x+1)(x^2-2x+3)} = -\dfrac{1}{2}\cdot\dfrac{1}{x+1} + \dfrac{3}{2}\cdot\dfrac{x-1}{x^2-2x+3}$.

EXERCISES Rewrite each of the following as a sum of simpler fractions.

1. $\dfrac{4}{(x+1)(x-5)}$ **2.** $\dfrac{2x-3}{x^2-5x+6}$ **3.** $\dfrac{5x}{2x^2+11x+12}$

4. $\dfrac{5x+1}{(x+3)(x+2)(x-4)}$ **5.** $\dfrac{4x^2-x+3}{(x+5)(x-1)(x-2)}$ **6.** $\dfrac{2x^2-5}{x^3-2x^2-3x}$

7. $\dfrac{x+2}{x^2+8x+16}$ **8.** $\dfrac{5-4x}{x^3+10x^2+25x}$ **9.** $\dfrac{5x^2-9x}{(x-4)(x-1)^2}$

10. $\dfrac{x^2-2}{(x+1)(x^2+3)}$ **11.** $\dfrac{3x^2+9x-4}{(x-1)(x^2+4x-1)}$ **12.** $\dfrac{6}{(x-5)(x^2-2x+3)}$

SOLUTIONS

1. $\dfrac{4}{(x+1)(x-5)} = \dfrac{A}{x+1} + \dfrac{B}{x-5} \quad\rightarrow\quad 4 = A(x-5) + B(x+1)$ for all x

$x=5 \rightarrow B=\frac{2}{3}$ and $x=-1 \rightarrow A=-\frac{2}{3}$

$$\frac{4}{(x+1)(x-5)} = \frac{2}{3}\cdot\frac{1}{x-5} - \frac{2}{3}\cdot\frac{1}{x+1}$$

2. $\dfrac{2x-3}{x^2-5x+6} = \dfrac{2x-3}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3} \quad\rightarrow\quad 2x-3 = A(x-3) + B(x-2)$ for all x

$x=3 \rightarrow B=3$ and $x=2 \rightarrow A=-1$

$$\frac{2x-3}{x^2-5x+6} = \frac{3}{x-3} - \frac{1}{x-2}$$

3. $\dfrac{5x}{2x^2+11x+12} = \dfrac{5x}{(2x+3)(x+4)} = \dfrac{A}{2x+3} + \dfrac{B}{x+4} \quad \rightarrow \quad 5x = A(x+4) + B(2x+3)$

$x = -4 \;\rightarrow\; B = 4$ and $x = -\frac{3}{2} \;\rightarrow\; A = -3$

$$\frac{5x}{2x^2+11x+12} = \frac{4}{x+4} - \frac{3}{2x+3}$$

4. $\dfrac{5x+1}{(x+3)(x+2)(x-4)} = \dfrac{A}{x+3} + \dfrac{B}{x+2} + \dfrac{C}{x-4}$

$\rightarrow \quad 5x+1 = A(x+2)(x-4) + B(x+3)(x-4) + C(x+3)(x+2)$

$x = -2 \;\rightarrow\; B = \frac{3}{2}$ and $x = 4 \;\rightarrow\; C = \frac{1}{2}$ and $x = -3 \;\rightarrow\; A = -2$

$$\frac{5x+1}{(x+3)(x+2)(x-4)} = -\frac{2}{x+3} + \frac{3}{2}\cdot\frac{B}{x+2} + \frac{1}{2}\cdot\frac{C}{x-4}$$

5. $\dfrac{4x^2-x+3}{(x+5)(x-1)(x-2)} = \dfrac{A}{x+5} + \dfrac{B}{x-1} + \dfrac{C}{x-2}$

$\rightarrow \quad 4x^2-x+3 = A(x-1)(x-2) + B(x+5)(x-2) + C(x+5)(x-1)$

$x = 1 \;\rightarrow\; B = -1$ and $x = 2 \;\rightarrow\; C = \frac{17}{7}$ and $x = -5 \;\rightarrow\; A = \frac{18}{7}$

$$\frac{4x^2-x+3}{(x+5)(x-1)(x-2)} = \frac{18}{7}\cdot\frac{1}{x+5} - \frac{1}{x-1} + \frac{17}{7}\cdot\frac{1}{x-2}$$

8. $\dfrac{5-4x}{x^3+10x^2+25x} = \dfrac{5-4x}{x(x+5)^2} = \dfrac{A}{x} + \dfrac{B}{x+5} + \dfrac{C}{(x+5)^2}$

$\rightarrow \quad 5-4x = A(x+5)^2 + Bx(x+5) + Cx$

$x=-5 \rightarrow C=-5$ and $x=0 \rightarrow A=\frac{1}{5}$ and $x=1,\ A=\frac{1}{5},\ C=-5 \rightarrow B=-\frac{1}{5}$

$$\frac{5\quad 4}{{}^{3}\quad 10\quad {}^{2}\quad 25}\qquad \frac{1}{5}\ \frac{1}{\quad 3}\qquad \frac{1}{5}\ \frac{1}{\quad 5}\qquad \frac{5}{(\quad +5)^2}$$

PART B:

Integration of proper Rational Expressions by Partial Fractions

In this part the student is expected to understand partial fraction decomposition as explained in Part A. The student is also expected to be able to perform elementary integrations (by substitution or by inspection) of the following types.

$$\int \frac{1}{ax+b}\,dx \;=\; \frac{1}{a}\cdot \ln\left| ax+b \right| \;+\; C$$

$$\int \frac{1}{(ax+b)^2}\,dx \;=\; -\frac{1}{a}\cdot\frac{1}{ax+b} \;+\; C$$

$$\int \frac{2ax+b}{ax^2+bx+c}\,dx \;=\; \ln\left| ax^2+bx+c \right| \;+\; C$$

In Part B each indefinite integral (antiderivative) must be simplified by decomposing the proper rational expression into a sum of partial fractions. The details of the partial fraction decomposition are left to the student. If necessary go back and review Part A.

Example 1 Find the indefinite integral. $\displaystyle\int \frac{7x+1}{(x+3)(x-1)}\,dx$

$(\quad)(\quad)$

$$\int \frac{x-3}{(x+2)(x^2+6)}\,dx = \int \left(-\frac{1}{2}\cdot\frac{1}{x+2} + \frac{1}{2}\cdot\frac{x}{x^2+6} \right) dx = -\frac{1}{2}\ln\left| x+2 \right| + \frac{1}{4}\ln\left| x^2+6 \right| + C$$

Example 5 Find the indefinite integral. $\displaystyle\int \frac{x^2+4x-7}{(x-3)(x^2+6x-6)}\,dx$

$$\int \frac{x^2+4x-7}{(x-3)(x^2+6x-6)}\,dx = \int \left(-\cdot\frac{1}{x-3} + -\cdot\frac{x+3}{x^2+6x-6} \right) dx = -\ln\left| x-3 \right| + \frac{}{6C}\ln\left| x^2+6x-6 \right| + C$$

5. $\int -\frac{1}{2}\cdot\frac{1}{x+2}+\frac{7}{6}\cdot\frac{1}{x-2}-\frac{14}{3}\cdot\frac{1}{x-5}\ dx = -\frac{1}{2}\cdot\ln|x+2| + \frac{7}{6}\cdot\ln|x-2| - \frac{14}{3}\cdot\ln|x-5| + C$

6.

Example 2 Find the indefinite integral. $\int \frac{\quad -}{\quad + \quad -}$

9. $\displaystyle\int \frac{x^5-2x^4-7x^3+20x^2-12x+4}{x^3+x^2-6x}\,dx$

10. $\displaystyle\int \frac{4x^5+6x^4+2x^3+3x^2-5x-7}{x^3+2x^2+x}\,dx$

11. $\displaystyle\int \frac{2x^3-15x^2+17x+25}{x^3-10x^2+25x}\,dx$

12. $\displaystyle\int \frac{+\quad - \ x \ - \ x+}{x \ + \ x \ + \ x}\,dx$

8. $\int \frac{x^4-10x^3+28x^2-15x-15}{x^3-7x^2+10x}\,dx = \int \; x-3+\frac{-3x^2+15x-15}{x(x-2)(x-5)} \; dx$

$= \int \; x-3-\frac{3}{2}\cdot\frac{1}{x}-\frac{1}{2}\cdot\frac{1}{x-2}-\frac{1}{x-5} \; dx = \frac{1}{2}x^2-3x-\frac{3}{2}\ln|x|-\frac{1}{2}\ln|x-2|-\ln|x-5|+C$

9. $\int \frac{x^5-2x^4-7x^3+20x^2-12x+4}{x^3+x^2-6x}\,dx = \int \; x^2-3x+2+\frac{4}{x(x+3)(x-2)} \; dx$

$= \int \; x^2 - 3x + 2 - \frac{2}{3}\cdot\frac{1}{x} + \frac{4}{15}\cdot\frac{1}{x+3} + \frac{2}{5}\cdot\frac{1}{x-2} \; dx$

$= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x - \frac{2}{3}\ln|x| + \frac{4}{15}\ln|x+3| + \frac{2}{5}\ln|x-2| + C$

10. $\int \frac{4x^5+6x^4+2x^3+3x^2-5x-7}{x^3+2x^2+x}\,dx = \int \; 4x^2-2x+2+\frac{x^2-7x-7}{x(x+1)^2} \; dx$

$= \int \; 4x^2 - 2x + 2 - \frac{7}{x} + \frac{8}{x+1} - \frac{1}{(x+1)^2} \; dx$

$= \frac{4}{3}x^3 - x^2 + 2x - 7\ln|x| + 8\ln|x+1| + \frac{1}{x+1} + C$

11. $\int \frac{2x^3-15x^2+17x+25}{x^3-10x^2+25x}\,dx = \int \; 2+\frac{5x^2-33x+25}{x(x-5)^2} \; dx = \int \; 2+\frac{1}{x}+\frac{4}{x-5}-\frac{3}{(x-5)^2} \; dx$

$= 2x + \ln|x| + 4\ln|x-5| - \frac{3}{x-5} + C$

12. $\int \frac{x^4+2x^3-6x^2-6x+3}{x^3+4x^2+4x}\,dx = \int \; x-2+\frac{-2x^2+2x+3}{x(x+2)^2} \; dx$

$= \int \; x - 2 + \frac{}{}\cdot\frac{1}{x} - \frac{}{5}\cdot\frac{1}{x+2} + \frac{}{}\cdot\frac{1}{(x+2)^2} \; dx$