# 201-NYA-05
# Calculus 1 (Science)
# Final Exam

May 16, 2011

[redacted]

Name: SOLUTIONS

Student ID: ______________

- There are a total of **100 marks** on this test.

[redacted]

1. [4+4+4 marks] Find each of the following limits. Please show all your work and give exact answers (no decimals).

(a) $\lim \dfrac{x^2 - 9x + 14}{\ldots}$ (b) $\lim \dfrac{\sqrt{x^2+48} - 7}{\ldots}$, (c) $\lim \dfrac{17x^5 - 3x^8}{\ldots}$

a) $$\lim_{x \to 2} \frac{x^2 - 9x + 14}{3x^2 - 6x} = \lim_{x \to 2} \frac{(x-7)\cancel{(x-2)}}{3x\cancel{(x-2)}} = \frac{2-7}{3(2)} = -\frac{5}{6}$$

or

$$\lim_{x \to 2} \frac{x^2 - 9x + 14}{3x^2 - 6x} = \lim_{x \to 2} \frac{2x - 9}{6x - 6} = \frac{4-9}{12-6} = -\frac{5}{6}$$

b) $$\lim_{x \to 1} \frac{\sqrt{x^2+48} - 7}{x-1} \cdot \frac{\sqrt{x^2+48} + 7}{\sqrt{x^2+48} + 7} = \lim_{x \to 1} \frac{(x^2+48) - 49}{(x-1)(\sqrt{x^2+48} + 7)}$$

$$= \lim_{x \to 1} \frac{x^2 - 1}{(x-1)(\sqrt{x^2+48} + 7)} = \lim_{x \to 1} \frac{x+1}{\sqrt{x^2+48} + 7} = \frac{1+1}{\sqrt{49} + 7}$$

$$= \frac{2}{14} = \frac{1}{7}$$

or

2. **[5 marks]** Find the values of $x$ where the following function is discontinuous. For each value, clearly state *why* $f(x)$ is discontinuous by using the definition of continuity at a point.

$$f(x) = \begin{cases} 1/x & \text{if } x < 1 \\ x & \text{if } 1 \le x < 2 \\ 3x^2 - 11x & \text{if } x \ge 2 \end{cases}$$

$\underline{x=0}$ : $f(0)$ DNE $\Longrightarrow$ $f(x)$ discontinuous at $x=0$

$\underline{x=2}$ : $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} x = 2$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} 3x^2 - 11x = 3(4) - 11(2)$

$= 12 - 22 = -10$

$\lim\limits_{x\to 2} f(x)$ DNE $\Longrightarrow$ $f(x)$ discontinuous at $x=2$.

3. [4+4+4+4 marks] Find the first derivative of each function. Answers do not need to be simplified but should be expressed only in terms of $x$

(a) $y = \tan(\sin x) + \tan x \sin x$

$$y' = \sec^2(\sin x) \cdot \cos x + \left[(\sec^2 x)\sin x + \tan x(\cos x)\right]$$

(b) $y = \dfrac{\ln(x^2+x+1)}{x^2+x+1}$

$$y' = \frac{\frac{1}{x^2+x+1}(2x+1)(x^2+x+1) - \ln(x^2+x+1)\cdot(2x+1)}{(x^2+x+1)^2}$$

(c) $y = \cot^2(3x^2 + 5x)$

$$y = \left(\cot(3x^2+5x)\right)^2$$

[redacted]

(d) $y = (\sin x)^{3x}$

$$\ln y = \ln\left((\sin x)^{3x}\right)$$

$$\ln y = 3x \ln(\sin x)$$

$$\frac{1}{y} y' = 3 \ln(\sin x) + 3x \cdot \frac{1}{\sin x}(\cos x)$$

[redacted]

4. **[5 marks]** Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \frac{2}{x}$. No marks will be given for using derivative rules.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$$

$$= \lim_{h \to 0} \frac{2(x) - 2(x+h)}{(x+h)(x)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x} - \cancel{2x} - 2h}{xh(x+h)}$$

$$= \lim_{h \to 0} \frac{-2}{x \cdot (x+h)}$$

$$= \frac{-2}{x^2}$$

5. **[4 marks]** Find the slope of the tangent line to the curve $x^2y - xy^2 = 6$ at the point $(3, 1)$.

$$x^2y - xy^2 = 6$$

$$(2xy + x^2y') - (1 \cdot y^2 + x \cdot 2yy') = 0$$

$$2(3)(1) + (3)^2y' - (1)(1)^2 - (3)(2)(1)y' = 0$$

$$3y' = -5$$

$$y' = -5/3$$

6. **[4+2 marks]** Suppose $f(x) = \arcsin 2x - \sqrt{1-4x^2}$.

(a) Find $f'(x)$ and simplify.

(b) Find $f''(0)$.

a) $f(x) = \arcsin(2x) - (1-4x^2)^{1/2}$

$$f'(x) = \frac{1}{\sqrt{1-(2x)^2}}(2) - \frac{1}{2}(1-4x^2)^{-1/2}(-8x)$$

$$= \frac{2}{\sqrt{1-4x^2}} + \frac{8x}{2\sqrt{1-4x^2}}$$

$$= \frac{2+4x}{\sqrt{1-4x^2}}$$

b) $f''(x) = (4)(1-4x^2)^{1/2} - (2+4x)\frac{1}{2}(1-4x^2)^{-1/2}(-8x)$

$$f''(0) = \frac{(4)(1)^{1/2} - (2)(\frac{1}{2})(1)^{-1/2}(0)}{1-4(0)^2}$$

$$= \frac{4-0}{1}$$

$$= 4$$

7. **[4+3 marks]** Consider the function $g(x) = (3x-4)^4(4x-3)^3$.

(a) Find $g'(x)$ and simplify.

(b) Find the absolute maximum and minimum values of $g(x)$ on the interval $[0, 1]$.

a) $$g'(x) = 4(3x-4)^3(3)(4x-3)^3 + 3(4x-3)^2(4)(3x-4)^4$$
$$= 12(3x-4)^3(4x-3)^2\left[(4x-3)+(3x-4)\right]$$
$$= 12(3x-4)^3(4x-3)^2(7x-7)$$

b) Critical numbers:
$$0 = 12(3x-4)^3(4x-3)^2(7x-7)$$
$$\Longrightarrow \quad x = \frac{4}{3}, \frac{3}{4}, 1$$

On the interval $[0,1]$, we check $x = 0, \frac{3}{4}, 1$ only:

$$f(0) = (0-4)^4(0-3)^3 = (256)(-27) = -6912$$
$$f(\tfrac{3}{4}) = (\tfrac{9}{4}-4)^4(3-3)^3 = 0.$$
$$f(1) = (3-4)^4(4-3)^3 = (-1)^4(1)^3 = 1$$

Absolute maximum : 1

Absolute minimum : $-6912$

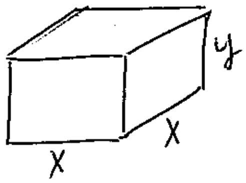
8. **[5 marks]** Boyles Law states that the product of the pressure exerted by a gas, $P$, and its volume, $V$, are related by $PV = k$ where $k$ is a constant. At a point in time, the [illegible]

gas is 2400 $cm^3$ and the pressure it exerts is 400 kPa. Find the [illegible]

if the volume is decreasing at a rate of 70 [illegible]

9. [**7 marks**] A box with a square base is to be constructed with volume 20 m$^3$. The material for the base of the box costs \$0.30/m$^2$, the material for the sides costs \$0.10/m$^2$, and the

material for the top of the box costs \$0.20/m$^2$. Find the dimensions that minimize the cost of the box (answer with the appropriate units).




Minimize cost:

$$C = (0.30)x^2 + (0.10)(4xy) + (0.20)x^2$$

$$C = 0.5x^2 + 0.4xy$$

Since $V = x^2 y \implies y = \dfrac{20}{x^2}$

$$C(x) = 0.5x^2 + 0.4x\left(\frac{20}{x^2}\right)$$

$$= 0.5x^2 + \frac{8}{x}$$

$$\implies C'(x) = x - \frac{8}{x^2}$$

$$0 = x - \frac{8}{x^2}$$

$$\frac{8}{x^2} = x$$

$$x^3 = 8$$

$$x = 2$$

The dimensions are

$x = 2$m, $y = \dfrac{20}{(2)^2} = 5$m.

Check max/min:

$$C''(x) = 1 + \frac{16}{x^3}$$

$> 0$ ∴ min at $x = 2$

10. **[2+3+3+2 marks]** Consider the following function and its derivatives:

$$f(x) = \frac{4(1-x)}{x^2} \qquad f'(x) = \frac{4(x-2)}{x^3} \qquad f''(x) = \frac{-8(x-3)}{x^4}$$

(a) Find the vertical and horizontal asymptotes, if they exist.

(b) Find the intervals of increase & decrease and local maxima & minima, if they exist.

(c) Find the intervals of concavity and inflection points, if they exist.

(d) Sketch the graph of $f(x)$.

NOTE: A grid and extra space may be found on the next page.

a) V. asymptotes: $\lim_{x\to 0} \frac{4(1-x)}{x^2} = \infty \quad \left(\frac{4}{0^+}\right)$

H. asymptote: $\lim_{x\to \pm\infty} \frac{4-4x}{x^2} = \lim_{x\to\pm\infty} \frac{4}{x^2} - \frac{4}{x} = 0$

$\Rightarrow$ VA at $x=0$, HA at $y=0$

b) $4(x-2)$ $- - - - \underset{2}{|} + +$

$x^3$ $- - \underset{0}{|} + + + + +$

$f'(x)$ $+ \underset{0}{|} - \underset{2}{|} +$

Increasing on $(-\infty, 0) \cup (2, \infty)$

Decreasing on $(0, 2)$

$f(2) = \frac{4(1-2)}{(2)^2} = -1 \Rightarrow$ local minimum at $(2, -1)$

no local maximum

Extra space for Question #10.

$$f(x) = \frac{4(1-x)}{x^2} \qquad f'(x) = \frac{4(x-2)}{x^3} \qquad f''(x) = \frac{-8(x-3)}{x^4}$$

c) $-8(x-3)$ : $+ + + + + \,|_{3}\, - -$

$x^4$ : $+ + \,|_{0}\, + + + + +$

$f''(x)$ : $+ \,|_{0}\, + \,|_{3}\, -$

Concave up: $(-\infty, 0) \cup (0, 3)$

Concave down: $(3, \infty)$

11. [4+4 **marks**] Find each limit:

(a) $\lim_{x\to\infty} x^3 e^{-x^2}$

(b) $\lim_{x\to 0} (1-x^2)^{1/x^2}$

a) $\lim_{x\to\infty} x^3 e^{-x^2}$ $(\infty \cdot 0)$

$= \lim_{x\to\infty} \frac{x^3}{e^{x^2}}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{3x^2}{2x e^{x^2}}$

$= \lim_{x\to\infty} \frac{3x}{2e^{x^2}}$

$= \lim_{x\to\infty} \frac{3}{2(2x)e^{x^2}}$

$= 0.$

b) $L = \lim_{x\to 0} (1-x^2)^{1/x^2}$

$\ln L = \lim_{x\to 0} \ln(1-x^2)^{1/x^2}$

$= \lim_{x\to 0} \frac{\ln(1-x^2)}{x^2}$

$= \lim_{x\to 0} \frac{\frac{1}{1-x^2}(-2x)}{2x}$

$= \frac{-1}{1-0} = -1$

$\Longrightarrow L = e^{-1} = \frac{1}{e}$

12. [**3 marks**] The velocity of a particle (in m/s) is given by the function $v(t) = \sin t - 4\sqrt{t}$. Find the position function $s(t)$ if $s(0) = 10$ m.

$$v(t) = \sin t - 4t^{1/2}$$

$$= -\cos t - \frac{8}{3}t^{3/2} + C$$

$$10 = -\cos(0) - \frac{8}{3}(0)^{3/2} + C$$

$$10 = -1 - 0 + C$$

$$\Rightarrow C = 11$$

13. [**4 marks**] Solve the following separable differential equation:

$$(1+x^2)\frac{dy}{dx} = 2y$$

$$\int \frac{1}{y}\,dy = \int \frac{2}{1+x^2}\,dx$$

$$\ln|y| = 2\arctan x + C$$

14. **[4+4 marks]** Evaluate each integral.

(a) $\int (x^3 + 6x^2 - 4)^8 (6x^2 + 24x)\, dx$ (b) $\int \frac{5\cos(\ln x)}{3x}\, dx$

a) $u = x^3 + 6x^2 - 4$

$du = (3x^2 + 12x)\,dx$

$2\,du = (6x^2 + 24x)\,dx$

$$\int (x^3+6x^2-4)^8 (6x^2+24x)\,dx = \int u^8 \cdot 2\,du$$
$$= 2\int u^8\,du$$
$$= \frac{2}{9}u^9 + C$$
$$= \frac{2}{9}(x^3+6x^2-4)^9 + C$$

b) $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \frac{5\cos(\ln x)}{3x}\,dx = \frac{5}{3}\int \cos(\ln x)\,\frac{1}{x}\,dx$$
$$= \frac{5}{3}\int \cos u\,du$$
$$= \frac{5}{3}\sin u + C$$
$$= \frac{5}{3}\sin(\ln x) + C$$